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Stability properties of Potts neural networks with biased patterns and low loading

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Abstract. The q-state Potts glass model of neural networks is extended to include biased patterns. For a finite number of such patterns, the existence and stability properties of the Mattis states and symmetric states are discussed in detail as a function of the bias. Analytic results are presented for all q at zero temperature. For finite temperatures numerical results are obtained for $q = 3$ and two classes of representative bias parameters. A comparison is made with the Hopfield model.

1. Introduction

The q-state Potts model [1] has been introduced in the theory of neural networks in [2] to include discrete neurons with more than two states. In that paper the capacity of storage and retrieval of information has been discussed, mostly concentrating on the limit of zero temperature.

A related model is the q-state clock spin glass [3] that has been studied as a neural network in [4]. In particular the phase diagram and storage capacity have been calculated and the information content has been considered for $q = 2, 3, 4$ and $q \to \infty$. In the same spirit, discrete-state phasor neural networks [5] are treated and their recall behaviour is solved exactly for any q for sparse and asymmetric interactions.

Another class of q-state generalizations of the Hopfield networks are the 3-state nets discussed in [6–8]. Recently [9] 2-state representations of such 3-state nets have been derived discussing to what extent the dynamical behaviour of the latter can be realized using 2-state neurons. 2-state representations of Potts glass models have been investigated in [10].

The q-state Potts and clock spin glass models for neural networks mentioned above concern independent random patterns taking any value in the set $\{1, \ldots, q\}$ with equal probability. In this work we extend these models to allow for the storage and retrieval of so-called biased patterns [11] with a probability $(1 + B_\alpha)/q$, $\alpha = 1, \ldots, q$, with B_α the bias parameter. In particular we are interested here in the stability properties of these networks at zero and finite temperatures as a function of the bias for a finite number of patterns. In a forthcoming publication we will discuss these type of networks near saturation, i.e. when the number of learned patterns increase with the size of the network [12].

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The rest of this paper is organized as follows. In section 2 we describe the model in detail. In section 3 we write down the free energy and the saddle-point equations for the overlap parameter, using the mean-field theory approach. Also the stability matrix and its eigenvalues are given. We then discuss in detail in section 4 the stability of the Mattis states and the symmetric states at zero temperature for general bias and arbitrary number of Potts states. In particular we give a set of rules for the bias parameters under which stability is guaranteed. Furthermore in section 5 we discuss the stability properties of the Mattis states and lowest symmetric states at finite temperature for $q = 3$ and two representative classes of bias parameters. We compare our results with the Hopfield model with biased patterns [11]. One of the findings is that for the $q = 3$ systems the stability region for the symmetric mixture states as a function of the temperature and bias is smaller, in comparison with the stability region for the Mattis state, than for the Hopfield model. In that sense we can say that these $q = 3$ systems perform better. Further, for a given bias some of the stability regions are disconnected as a function of the temperature.

A summary of the main results is given in section 6. Finally in the appendix we work out some details about the behaviour of the eigenvalues of the stability matrix in the limit of zero temperature.

2. The model

Consider a system of N neurons. We assume that every neuron can occupy q discrete states by viewing it as a q-state Potts spin. The instantaneous configuration of all the spin variables at a given time describes the state of such a network. The neurons are interconnected with all the others by a synaptic matrix of strength $J_{ij}^{\alpha\rho}$ which determines the contributions of a signal fired by the jth presynaptic neuron in state ρ to the post-synaptic potential which acts on the ith neuron in state α. This contribution can either be positive (excitatory synapse) or negative (inhibitory synapse).

The potential h_{σ_i} of neuron i which is in a state σ_i is the sum of all postsynaptic potentials delivered to it in a time unit, i.e.

$$h_{i,\sigma_i} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{\alpha,\rho=1}^{q} J_{ij}^{\alpha\rho} m_{\sigma_i,\alpha} m_{\sigma_j,\rho} \tag{2.1}$$

with

$$m_{\sigma_i,\rho} = q\delta_{\sigma_i,\rho} - 1. \tag{2.2}$$

We assume that the synaptic couplings satisfy

$$J_{ij}^{\alpha\rho} = J_{ji}^{\rho\alpha}. \tag{2.3}$$

The dynamics of this q-state Potts model is the following. At zero temperature the state of the neuron in the next time step is fixed to be the state which minimizes the induced local field (2.1). The stable states of the system are those configurations where every neuron is in a state which gives a minimum value to $\{h_{i,\sigma_i}\}$. If the relation (2.3) holds, this stability is equivalent to the requirement that the configurations $\{\sigma_i\}$ are the local minima of the anisotropic Hamiltonian

$$H = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \sum_{\alpha,\rho=1}^{q} J_{ij}^{\alpha\rho} m_{\sigma_i,\alpha} m_{\sigma_j,\rho}. \tag{2.4}$$

In the presence of noise there is a finite probability of having configurations other than the local minima. This can be taken into account by introducing an effective temperature $T = 1/\beta$.

To build in the capacity of learning and memory in this network, its stable configurations must be correlated with the p patterns $\{k_i^a\}$, $a = 1, \ldots, p$ fixed by the learning process. The latter are allowed to be biased, i.e. the k_i^a are chosen as independent random variables which can take the values $1, \ldots, q$ with probability

$$P(\alpha) = \frac{1 + B_\alpha}{q} \qquad \alpha = 1, \ldots, q \tag{2.5}$$

where the $\{B_\alpha\}$ are the bias parameters. Analogous to the Hopfield model [11], we therefore propose the learning rule

$$J_{ij}^{\alpha\rho} = \frac{1}{q^2 N} \sum_{a=1}^{p} \left(m_{k_i^a,\alpha} - B_\alpha \right) \left(m_{k_j^a,\rho} - B_\rho \right). \tag{2.6}$$

We note that the bias parameters are independent of the patterns and hence the latter are learned in the same way. From the fact that $0 \leq P(\alpha) \leq 1$ and $\sum_{\alpha=1}^{q} P(\alpha) = 1$ we deduce the properties

$$-1 \leq B_\alpha \leq q - 1 \qquad \sum_{\alpha=1}^{q} B_\alpha = 0. \tag{2.7}$$

We further remark that the biased Hebb rule (2.6) satisfies the following properties. Setting all $B_\alpha = 0$, $\alpha = 1, \ldots, q$, leads to the q-state Potts model discussed in [2]. As in the Hopfield model [11] we find that

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{\rho=1}^{q} \Delta J_{ij}^{\alpha\rho} = 0 \qquad \Delta J_{ij}^{\alpha\rho} = J_{ij}^{\alpha\rho}(p+1) - J_{ij}^{\alpha\rho}(p) \tag{2.8}$$

i.e. the total modification of synapses on a given neuron is unchanged during learning.

Using the learning rule (2.6), both the postsynaptic potential and the Hamiltonian can be rewritten as

$$h_{i,\sigma_i} = -\frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{a=1}^{p} \left(m_{k_i^a,\sigma_i} - B_{\sigma_i} \right) \left(m_{k_j^a,\sigma_j} - B_{\sigma_j} \right) \tag{2.9}$$

$$H = -\frac{1}{2N} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \sum_{a=1}^{p} \left(m_{k_i^a,\sigma_i} - B_{\sigma_i} \right) \left(m_{k_j^a,\sigma_j} - B_{\sigma_j} \right) \tag{2.10}$$

where we have employed ($\rho, \gamma = 1, ..., q$)

$$\frac{1}{q}\sum_{\alpha=1}^{q} m_{\rho,\alpha} m_{\alpha,\gamma} = m_{\rho,\gamma}. \tag{2.11}$$

In the following we study the biased Potts neural network for finite p and q and in the limit $N \to \infty$. An important role is played by the order parameter R_a defined by

$$R_a = \frac{1}{N}\sum_{j=1}^{N}\left(m_{\sigma_j,k_j^a} - B_{\sigma_j}\right). \tag{2.12}$$

It can be interpreted as a measure for the macroscopic overlap with pattern a. When we have total overlap, R_a becomes in the limit $N \to \infty$

$$R_a = q - 1 - \frac{1}{q}\sum_{\alpha=1}^{q} B_\alpha^2. \tag{2.13}$$

For random configurations with probability distribution (2.5), we have

$$R_a = -\frac{1}{q}\sum_{\alpha=1}^{q} B_\alpha^2. \tag{2.14}$$

3. Mean-field theory for low loading

Starting from the Hamiltonian (2.10) and applying standard techniques (linearization and the saddle-point method [11, 13]), the ensemble-averaged free energy is given by

$$f = \frac{1}{2}\sum_{a=1}^{p} R_a^2 - \frac{1}{\beta}\left\langle\!\!\left\langle \ln\left(\sum_{\rho=1}^{q}\exp\left[\beta\sum_{a=1}^{p}(m_{k^a,\rho} - B_\rho)R_a\right]\right)\right\rangle\!\!\right\rangle. \tag{3.1}$$

The double brackets $\langle\langle . \rangle\rangle$ stand for averaging over the distribution of all learned patterns $\{k^a\}$. The saddle-point equations for the order parameters R_a are given by

$$R_a = \left\langle\!\!\left\langle \frac{\sum_{\rho=1}^{q}\left(m_{k^a,\rho} - B_\rho\right)\exp\left[\beta\sum_{b=1}^{p}(m_{k^b,\rho} - B_\rho)R_b\right]}{\sum_{\rho=1}^{q}\exp\left[\beta\sum_{b=1}^{p}(m_{k^b,\rho} - B_\rho)R_b\right]}\right\rangle\!\!\right\rangle. \tag{3.2}$$

The following type of solutions of (3.2) will be distinguished. First, there are the Mattis states, having only one non-zero overlap which we denote by M. If $M > 0$, they are correlated with one of the p learned patterns. Since all patterns are treated in the same way the index of the pattern can be chosen arbitrarily so that we have p solutions of this type.

Second, we have the n-symmetric states, being solutions with n ($1 \le n \le p$) non-zero overlaps with equal magnitude S_n. When $S_n > 0$, they can be seen as states which mix the n corresponding patterns. We remark that S_1 is the Mattis state M.

Third, there are the asymmetric states, having n $(1 < n \leq p)$ non-zero overlaps with different magnitude.

We are interested in the existence and stability properties of these solutions. As indicated above, from a neural network point of view the positive Mattis states are especially important. But of course we also want to know if, and in what temperature regime, the other type of states, which are spurious states, can be stable. Since, as we will see in the following sections, the study of these Potts networks is very involved, we have restricted ourselves in this paper to a treatment of the spurious symmetric states.

In order to discuss the local stability of the symmetric and Mattis states, we look at the stability matrix

$$A_{ab} \equiv \frac{\partial^2 f}{\partial R_a \partial R_b}. \tag{3.3}$$

This matrix is of the form

$$A = \begin{pmatrix} \mathcal{A}_n & 0 \\ 0 & \mathcal{D}_{p-n} \end{pmatrix} \tag{3.4}$$

where $\mathcal{A}_n$ is an $n \times n$ matrix with diagonal elements γ_1 and off-diagonal elements δ and $\mathcal{D}_{p-n}$ is a $(p-n) \times (p-n)$ diagonal matrix with elements γ_2. The quantities γ_1, γ_2 and δ are given by

$$\gamma_1 = 1 - \beta \left\langle\!\!\left\langle \frac{\sum_{\rho=1}^{q} \left(m_{k^1,\rho} - B_\rho\right)^2 U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} - \left(\frac{\sum_{\rho=1}^{q} \left(m_{k^1,\rho} - B_\rho\right) U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} \right)^2 \right\rangle\!\!\right\rangle \tag{3.5}$$

$$\gamma_2 = 1 - \beta \left\langle\!\!\left\langle \frac{\sum_{\rho=1}^{q} \left(m_{k^{n+1},\rho} - B_\rho\right)^2 U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} - \left(\frac{\sum_{\rho=1}^{q} \left(m_{k^{n+1},\rho} - B_\rho\right) U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} \right)^2 \right\rangle\!\!\right\rangle \tag{3.6}$$

$$\delta = -\beta \left\langle\!\!\left\langle \frac{\sum_{\rho=1}^{q} \left(m_{k^1,\rho} - B_\rho\right)\left(m_{k^2,\rho} - B_\rho\right) U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} - \frac{\left[\sum_{\rho=1}^{q} \left(m_{k^1,\rho} - B_\rho\right) U_\rho(n)\right]\left[\sum_{\rho=1}^{q} \left(m_{k^2,\rho} - B_\rho\right) U_\rho(n)\right]}{\left(\sum_{\rho=1}^{q} U_\rho(n)\right)^2} \right\rangle\!\!\right\rangle \tag{3.7}$$

where

$$U_\rho(n) = \exp\left[\beta S_n \left(\sum_{s=1}^{n} m_{k^s,\rho} - nB_\rho\right)\right]. \tag{3.8}$$

The matrix A has three different eigenvalues:

$$\lambda_1 = \gamma_1 - \delta \tag{3.9}$$

$$\lambda_2 = \gamma_1 + (n-1)\delta \tag{3.10}$$

$$\lambda_3 = \gamma_2 \tag{3.11}$$

with respective degeneracy 1, $n-1$ and $p-n$. The signs of λ_1, λ_2 and λ_3 determine the stability of the solutions of (3.2).

For a Mattis state the saddle-point equation (3.2) can be written more explicitly as

$$M = \sum_{\alpha=1}^{q} \left(\frac{1+B_\alpha}{q}\right) \frac{\sum_{\rho=1}^{q}(m_{\alpha,\rho}-B_\rho)\exp\left[\beta M(m_{\alpha,\rho}-B_\rho)\right]}{\sum_{\rho=1}^{q}\exp\left[\beta M(m_{\alpha,\rho}-B_\rho)\right]}. \tag{3.12}$$

Furthermore, the explicit form of the matrix elements of the stability matrix (3.3) is given by

$$A_{ab} = \delta_{ab} - \beta \left\langle\!\!\left\langle \frac{\sum_{\rho=1}^{q}(m_{k^a,\rho}-B_\rho)(m_{k^b,\rho}-B_\rho)U_\rho}{\sum_{\rho=1}^{q}U_\alpha} - \frac{\sum_{\rho=1}^{q}\sum_{\alpha=1}^{q}(m_{k^a,\rho}-B_\rho)(m_{k^b,\alpha}-B_\alpha)U_\rho U_\alpha}{\left(\sum_{\rho=1}^{q}U_\rho\right)^2} \right\rangle\!\!\right\rangle \tag{3.13}$$

where now

$$U_\rho = \exp\left(\beta \sum_{d=1}^{p}(m_{k^d,\rho}-B_\rho)\,R_d\right). \tag{3.14}$$

It is clear that A is diagonal. In this case there are only two different eigenvalues: a non-degenerate one, namely

$$\lambda_1 = 1 - \beta\sum_{\alpha=1}^{q}\left(\frac{1+B_\alpha}{q}\right)\left[\frac{\sum_{\sigma=1}^{q}(m_{\sigma,\alpha}-B_\sigma)^2\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}{\sum_{\sigma=1}^{q}\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]} - \left(\frac{\sum_{\sigma=1}^{q}(m_{\sigma,\alpha}-B_\sigma)\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}{\sum_{\sigma=1}^{q}\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}\right)^2\right] \tag{3.15}$$

and a $(p-1)$-times degenerate one, i.e.

$$\lambda_3 = 1 - \beta\sum_{\alpha=1}^{q}\sum_{\rho=1}^{q}\left(\frac{1+B_\alpha}{q}\right)\left(\frac{1+B_\rho}{q}\right)\left[\frac{\sum_{\sigma=1}^{q}(m_{\sigma,\rho}-B_\sigma)^2\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}{\sum_{\sigma=1}^{q}\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]} - \left(\frac{\sum_{\sigma=1}^{q}(m_{\sigma,\rho}-B_\sigma)\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}{\sum_{\sigma=1}^{q}\exp[\beta M(m_{\sigma,\alpha}-B_\sigma)]}\right)^2\right]. \tag{3.16}$$

Again, the signs of λ_1 and λ_3 determine the stability of the solutions of (3.12).

In the next section we study in detail the free energy and the existence and stability of the solutions of the saddle-point equations (3.2) and (3.12) at zero temperature.

4. Results at zero temperature

We start by discussing the solutions of the fixed point equations (3.2) and (3.12) at $T=0$. In the course of this discussion it will be convenient to rewrite the bias $\boldsymbol{B}=(B_1,...,B_q)$ in a different form, namely

$$\boldsymbol{B} = a(b_1,...,b_q) \qquad b_1 \geq b_2 \geq ... \geq b_q, \;\; a \in (0,1]. \tag{4.1}$$

We call a the bias amplitude and $(b_1,...,b_q)$ the bias structure. Due to the fact that the model described in section 2 is invariant under permutations of the states of a neuron i, (4.1) is not an additional assumption.

4.1. The Mattis states

Taking the limit $\beta \to \infty$ of (3.12), it is straightforward to see that the following solutions exist:

$$M_0^+ = q - 1 - \frac{a^2}{q}\sum_{\alpha=1}^{q} b_\alpha^2 \tag{4.2}$$

$$M_0^- = -\frac{1+ab_1}{q}(q + ab_2 - ab_1). \tag{4.3}$$

We remark that M_0^- only depends on b_1 and b_2 because of the ordering (4.1) and the properties (2.7). The subscript 0 indicates that the solution is taken at $T = 0$ and the superscript $\pm$ expresses the fact that the solution is positive (respectively negative). We remark here that comparing (4.2) with (2.14) we see that we have complete retrieval since, as we will show now, M_0^+ is stable near $T = 0$.

To check the stability for the Mattis states we have to calculate the eigenvalues (3.15) and (3.16) in the limit $T \to 0$. This is straightforward but tedious, especially for the negative solution. For more details we refer to the appendix.

In particular for M_0^+ both eigenvalues λ_1 and λ_3 tend to one such that the positive Mattis states are always stable at $T = 0$ independent of the structure of the bias. For M_0^-, λ_1 tends to one but the behaviour of λ_3 depends on the structure of the bias parameters. More explicitly, the M_0^- are stable in the following cases:

- $p = 1$
- $p > 1$, $q = 2$ and $ab_1 \neq 1$
- $p > 1$, $q > 2$ and $b_1 \neq b_2 \neq b_3$
- $p > 1$, $q > 2$ and $ab_1 \neq q - 1$.

Otherwise they are unstable. We note that for the Potts model ($q \geq 3$) without bias the M_0^+ are stable and the M_0^-, except for $p = 1$, are unstable. The results mentioned above incorporate the known results about stability for $q = 2$ with and without bias [11,13]. Further in this section we will work out two classes of representative bias parameters explicitly for $q = 3$.

4.2. The symmetric states

From (3.2) we find in the limit $\beta \to \infty$

$$S_n^+ = \frac{1}{n}\left\langle\!\left\langle \max_\rho \left(\sum_{a=1}^{n} m_{k^a,\rho} - nab_\rho\right)\right\rangle\!\right\rangle \tag{4.4}$$

$$S_n^- = \frac{1}{n}\left\langle\!\left\langle \min_\rho \left(\sum_{a=1}^{n} m_{k^a,\rho} - nab_\rho\right)\right\rangle\!\right\rangle. \tag{4.5}$$

The study of their stability properties again requires a detailed investigation of the eigenvalues λ_1 to λ_3 (3.9)–(3.11) in the limit $\beta \to \infty$. Based on the general argumentation given in the appendix, the following results are obtained.

We first discuss the positive solutions. The even symmetric states S_{2l}^+ are unstable if there are at least two equal bias parameters. Otherwise, for a given bias structure, they are stable except for a finite number of values of the bias amplitude a. In the following, stability is always meant in this sense. The odd symmetric states S_{2l+1}^+ are unstable for $q \geq 3$ and at least two equal bias parameters for the values of the bias amplitude satisfying

$$(l-1)q \geq (2l+1)a(b_k - b_3)\delta_{k,1} \qquad lq \geq (2l+1)a(b_k - b_q) \tag{4.6}$$

where

$$k = \max\{j \in \{1, \dots, q-1\} | b_j = b_{j+1}\}. \tag{4.7}$$

Otherwise they are stable except for a finite number of values of a. For $q = 2$ we find the known results [11, 13]: the even symmetric states are unstable without bias and stable with bias; the odd symmetric states are stable for all values of the bias parameters.

Second, for completeness, we look at the negative solutions. If all bias parameters are different, they are stable. If there are at least two equal bias parameters we define k' as

$$k' = \min\{j \in \{1, \dots, q-1\} | b_j = b_{j+1}\}. \tag{4.8}$$

Two possibilities have to be considered. First, if $k' = 1$ the negative symmetric states are unstable for a given bias structure in the following cases:

- $q = 2$ and n even
- $\forall a \in (0, 1] : na(b_1 - b_{n+1}) \geq q$ if $1 < n = p < q$
- $\forall a \in (0, 1]$ if $q > 2$ and $n < p$
- $\forall a \in (0, 1]$ if $n = p \geq q > 2$.

Otherwise they are stable. Second, if $k' \geq 2$ they are unstable in the cases

- $\forall a \in (0, 1]$:

$$-nab_{k'} \leq \max_{d \in D_1} \left\{ \min_{j=1,\dots,k'-1} [d_j q - nab_j] \right\}$$

if $1 < n < p$ and $n \geq k' - 1$ with

$$D_1 = \left\{ d \in \mathbb{N}_0^{k'-1} \mid d_i \geq d_j \text{ if } 1 \leq i \leq j \leq k' - 1 \, ; \sum_{j=1}^{k'-1} d_j = n \right\} \tag{4.9}$$

- $\forall a \in (0, 1]$:

$$q - nab_{k'} \leq \max_{d \in D_2} \left\{ \min_{j=1,\dots,k'-1} [d_j q - nab_j] \right\}$$

if $2k' \leq n = p$ with

$$D_2 = \left\{ d \in (\mathbb{N} \backslash \{0, 1\})^{k'-1} \mid d_i \geq d_j \text{ if } 1 \leq i \leq j \leq k' - 1 \, ; \sum_{j=1}^{k'-1} d_j = n - 2 \right\}. \tag{4.10}$$

Otherwise they are stable. For the Potts model ($q \geq 3$) without bias all symmetric states are unstable except the negative ones with $n = p < q$. This has also been found in a study of the dynamics of these models [14, 15].

4.3. Specific $q = 3$ models

We now illustrate these results for $q = 3$ and two representative classes of bias parameters. We therefore choose

$$\boldsymbol{B}_1 = a(2,-1,-1) \tag{4.11}$$

$$\boldsymbol{B}_2 = a(1,0,-1) \tag{4.12}$$

with $a \in (0,1]$. The form (4.11) indicates that one state is privileged and the other two states have equal probability to appear. In the other case, (4.12), all three states have different probability. We recall that the biased Hopfield model has $\boldsymbol{B} = a(1,-1)$, $a \in (0,1]$.

The general stability results applied to these cases give the following. For $\boldsymbol{B}_1$, we find that the positive Mattis states are stable. The negative ones are stable for $p = 1$ and also stable for $p > 1$ if the bias amplitude is maximal, i.e. if $a = 1$. Furthermore, the positive symmetric states are unstable. The negative symmetric ones are unstable if $1 < n < p$ and if $1 < n = p$ they are unstable for $p \geq 4$ in the region for the bias amplitude given by $0 \leq a < 1 - 3/p$. Otherwise they are stable (up to a finite number of values for a).

For $\boldsymbol{B}_2$ the positive and negative Mattis states are always stable. The positive and negative symmetric states are also stable. Again stability is up to some values of a, e.g. for $n = 3$, a has to be different from $\frac{1}{2}$ and 1 for S_3^+ and S_3^-.

A conclusion of this stability analysis can be that biased $q = 3$ models, where two states have equal probability to appear, resemble the models without bias. They have less stable spurious symmetric states than models where all states have different probability.

4.4. The free energy

Taking the limit $\beta \to \infty$ of (3.1) we find, with obvious notation

$$f_n^{\pm}(T=0) = -\tfrac{1}{2} n \left(S_n^{\pm}\right)^2 \qquad n = 1, ..., p. \tag{4.13}$$

Without bias it has been shown for $q = 2$ [13] that the following order of energy levels of the positive symmetric states occurs at $T = 0$

$$f_1^+ < f_3^+ < ... < f_4^+ < f_2^+. \tag{4.14}$$

This relation is no longer true for $q \geq 3$. For example, we find that $f_4^+ > f_2^+$ for $q > 5$. For large q we even have that $S_n^+ \sim q/n$ such that

$$f_1^+ \leq f_2^+ \leq f_3^+ \leq \cdots. \tag{4.15}$$

In the case with bias we have for $q = 2$ that the Mattis states are no longer the ground states at $T = 0$ when the bias parameters exceeds the value $a = \sqrt{2} - 1$. For $q \geq 3$ these type of results are strongly dependent upon the specific bias parameters. For example, for (4.11) and (4.12) we arrive at the following results for the positive Mattis and the lowest positive symmetric energy levels. For $\boldsymbol{B}_1$

$$f_1^+ = -2\left(1 - a^2\right)^2 \tag{4.16}$$

$$f_2^+ = -\tfrac{1}{9}(1-a)^2\left(3 + 5a + 4a^2\right)^2 \tag{4.17}$$

$$f_3^+ = \begin{cases} -\frac{2}{27}(1-a)^4\left(4 + 9a + 8a^2\right)^2 & \text{if } a \leq \frac{1}{3} \\ -\frac{2}{27}(1-a)^2\left(3 + 4a + 7a^2 + 4a^3\right)^2 & \text{if } a \geq \frac{1}{3} \end{cases} \tag{4.18}$$

and for $\boldsymbol{B}_2$

$$f_1^+ = -\tfrac{2}{9}\left(3 - a^2\right)^2 \tag{4.19}$$

$$f_2^+ = -\tfrac{1}{81}\left(9 + 4a - 3a^2 - 2a^3\right)^2 \tag{4.20}$$

$$f_3^+ = \begin{cases} -\tfrac{2}{243}\left(12 + 3a - 6a^2 - 3a^3 + 2a^4\right)^2 & \text{if } a \le \tfrac{1}{2} \\ -\tfrac{1}{486}\left(21 + 9a - 3a^2 - 9a^3 - 2a^4\right)^2 & \text{if } a \ge \tfrac{1}{2}. \end{cases} \tag{4.21}$$

The different forms for f_3^+ arise because at the points $a = \frac{1}{3}$, $a = \frac{1}{2}$ the exponentials that contribute in the limit $\beta \to \infty$ change. The energy levels are shown in figures 1 and 2. We see again that for bias $\boldsymbol{B}_1$ the Mattis state and the first symmetric state cross when the bias parameter exceeds the value $a_0 = \frac{1}{8}\sqrt{18\sqrt{2} - 5} + 3\sqrt{2} - 5$. Furthermore, since as we have seen above, the symmetric states are unstable and the Mattis state is stable, there must be another solution of the fixed point (3.2) which is the global minimum of the free energy for $a > a_0$. This might be an asymmetric state. For bias $\boldsymbol{B}_2$ the Mattis state remains the lowest in energy for all values of the bias amplitude.

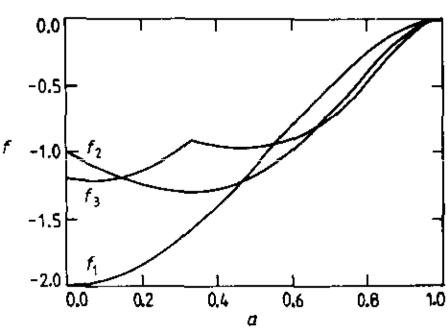

Figure 1. The energies of the first three positive symmetric states at $T = 0$ for the $q = 3$ Potts network with bias $\boldsymbol{B}_1 = a(2, -1, -1)$ as a function of the bias parameter a.

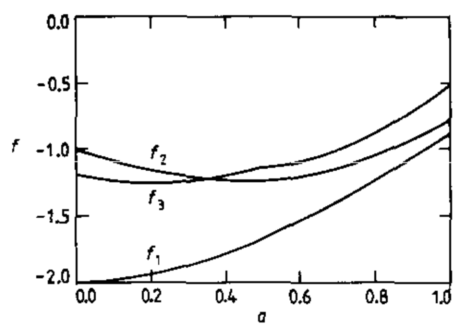

Figure 2. The energies of the first three positive symmetric states at $T = 0$ for the $q = 3$ Potts network with bias $\boldsymbol{B}_2 = a(1, 0, -1)$ as a function of the bias parameter a.

Finally, concerning the free energy levels for the negative Mattis states we find in general, combining (4.3) and (4.13)

$$f_1^- = -\frac{1}{2q^2}(1 + ab_1)[q + a(b_2 - b_1)]^2. \tag{4.22}$$

Using this and the properties of the bias parameters (2.7) for $q = 3$ we arrive, after some manipulations, at

$$f_1^- > f_1^+.$$

Therefore a negative Mattis state can not be a global minimum of the free energy.

5. Results at finite temperature

The stability study at finite temperature is much more involved. In the following we mainly restrict ourselves to positive states since they correspond to retrieval and mixture states. First we define the following temperatures which are of interest:

$$T_0 = \inf \{T | (T, \boldsymbol{R} = 0) \text{ is a stable solution of } (3.2)\} \tag{5.1}$$

$$T_n = \sup \{T | \exists S_n > 0 : (T, \boldsymbol{S}_n) \text{ is a stable solution of } (3.2)\} \tag{5.2}$$

where $\boldsymbol{R} = (R_1, ..., R_p)$ and $\boldsymbol{S}_n$ is a p-dimensional vector with the first n components equal to S_n and the other components equal to zero. We note that the critical temperature, i.e. the temperature below which the Mattis states become an absolute minima of the free energy, is important from a thermodynamic point of view but has no particular significance for neural networks. There the temperature T_1 below which the metastable retrieval states appear is significant.

We concentrate on the $q < 4$ models with and without bias. We study the stability properties in the whole temperature region from $T = 0$ up to $\max\{T_0, T_n\}$. The temperature T_0 can be found from

$$T_0 = \inf\{T | \gamma_1(T, \boldsymbol{R} = 0) > 0 \qquad \gamma_2(T, \boldsymbol{R} = 0) > 0\} \tag{5.3}$$

using the expressions (3.5) and (3.6). The result is

$$T_0 = q - 1 - \frac{a^2}{q} \sum_{\rho=1}^{q} b_\rho^2. \tag{5.4}$$

Above this temperature, the state $\boldsymbol{R} = 0$ becomes stable.

5.1. The Mattis states

The calculation of T_1 for the Mattis states is more difficult. For $q = 2$ some analytic results in this connection are obtained in [11]. In that case $T_0 = 1 - a^2$ and, using series expansions of (3.15) and (3.16) around T_0, one shows that the Mattis states exist and become unstable near T_0 if $a > 1/\sqrt{3}$.

We have supplemented these $q = 2$ stability results by determining precisely the stability region of the Mattis states as a function of T and a, over the whole interval of these parameters. The result is shown in figure 3. One sees that for $0 < a < 1/\sqrt{3}$, $T_1 = T_0$ while for $1/\sqrt{3} < a < 1$, $T_1 < T_0$. Furthermore there is a small interval for the bias amplitude $(0.54, 1/\sqrt{3})$ where, as one lowers the temperature starting from T_1, one encounters an instability region. Since we know that the Mattis states are stable at $T = 0$, we conclude that for this bias interval the temperature regions where we have stable Mattis states are disconnected.

A similar study has been done numerically for the $q = 3$ Mattis states and both classes of bias parameters (4.11) and (4.12). The results are shown in figures 4 and 5. For $a = 0$, we find the value of T_1 in the unbiased case [15]: $T_1(a = 0) = 2.18$. When a increases, T_1 decreases. For $a = 1$ we find a T_1 that is zero for the class $\boldsymbol{B}_1$ and non-zero for the class $\boldsymbol{B}_2$. The reason is the following: the bias $\boldsymbol{B}_1 = (2, -1, -1)$ corresponds with a probability distribution for the patterns where the lowest state has probability one. This means that there is no freedom left for the neurons. The

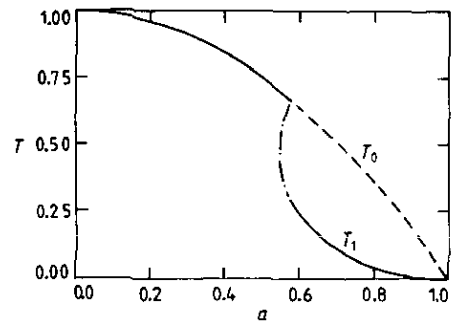

Figure 3. The temperatures T_0 and T_1 as a function of the bias for the Hopfield model. The chain curve indicates the border of the stability region.

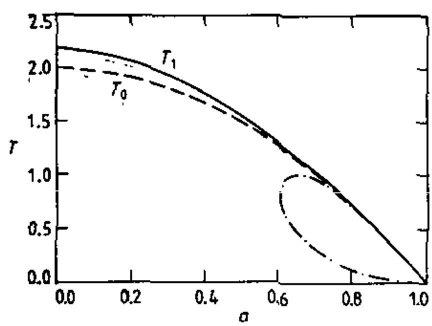

Figure 4. The temperatures T_0 and T_1 as a function of the bias for the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_1$ Potts network. The chain curve indicates the border of the stability region.

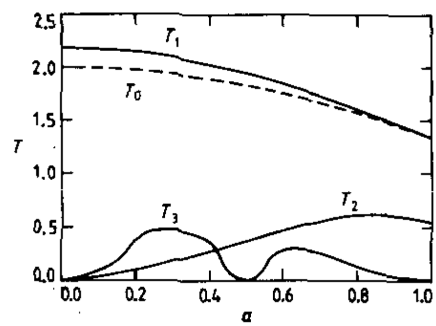

Figure 5. The temperatures T_0, T_1, T_2 and T_3 as a function of the bias for the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_2$ Potts network.

bias $\boldsymbol{B}_2 = (1, 0, -1)$ corresponds with a distribution where two states have a non-zero probability. Hence the neurons can still occupy different states.

Numerical evaluation of λ_1 (3.15) and λ_3 (3.16) give the precise stability results shown in the same figures. First, they confirm the preliminary results presented in [14], i.e. the Mattis states are no solution of the fixed-point equation (3.12) above T_1 for a bias of the form (4.11) and (4.12) independent of the value of a in contrast to the Hopfield model. Furthermore there is again an instability region for $\boldsymbol{B}_1$ in the interval $a \in [0.6, 1]$ just as for the Hopfield case. For $\boldsymbol{B}_2$ such a region does not appear.

Second, contrary to the Hopfield model, where always $T_1 \leq T_0$ there is a region (T_0, T_1) depending on a where both a Mattis state and the zero-solution are stable. For $a \to 1$, this region contracts to zero. In fact the situation in this region can be described more precisely as follows. Below T_0 only one (positive) Mattis solution exists and it is stable. At T_0 the zero solution becomes stable and the stable Mattis state has a finite overlap. In the interval (T_0, T_1) there exists, besides the stable zero and Mattis state, a second unstable Mattis state. The latter has overlap zero at T_0 and it increases with increasing temperature. In contrast, the overlap of the stable Mattis state decreases. At T_1 both Mattis states coalesce and disappear. The appearance of such a region is consistent with the study of the time evolution of the overlap presented in [15]. There it has been calculated that for $a = 0$, the overlap jumps from 0.75 to 0 at $T_1 = 2.18$. All this is reminiscent of a first-order phase transition at T_1 for the $q = 3$ models. Therefore, deriving analytic results based on the series expansions for small overlap, as has been done in the Hopfield case, does not make any sense in the neighbourhood of the interesting temperature T_1.

5.2. *The symmetric states*

Analytic results based on series expansions for small overlap S_n give information about what happens near T_0 in the following way. Starting from the expanded fixed-point equation (3.2) for the symmetric states

$$S_n^{\pm} = \beta(q - 1 - \frac{a^2}{q}\sum_{\rho=1}^{q} b_\rho^2)S_n^{\pm} + \beta^2\left[(q-1)(q-2) - \frac{3(q-2)a^2}{q}\sum_{\rho=1}^{q} b_\rho^2 + \frac{2a^3}{q}\sum_{\rho=1}^{q} b_\rho^3\right] \times \tfrac{1}{2}(S_n^{\pm})^2 + \mathrm{O}((\beta S_n^{\pm})^3) \tag{5.5}$$

we see that, on the one hand, for the Hopfield model the β^2 term is zero. Furthermore, the β^3 term for the positive symmetric states in that case ($q = 2$), i.e.

$$-(\beta S_n^{+})^3\left[(n-2)(a^2-1)\left(a^2 - \frac{3n-2}{3n-6}\right)\right] \tag{5.6}$$

is always negative. So, this immediately suggests that for $(1 - a^2)\beta < 1$, i.e. $T > T_0$, the only possible solution is $\boldsymbol{R} = 0$. On the other hand, for the $q \geq 3$ Potts models the second term in (5.5) can be positive (take e.g. $a = 0$). Hence, a positive overlap solution is possible for $\beta(q-1-(a^2/q)\sum_{\rho=1}^{q} b_\rho^2) < 1$, i.e. again $T > T_0$, and a negative overlap solution is possible for $T < T_0$.

Using these expansions for the $S_n^{\pm}$ in (3.9)–(3.11), we find for the eigenvalues λ_1 to λ_3

$$\lambda_1 \approx \lambda_2 \approx \frac{t}{T} + \mathrm{O}(t^2) \tag{5.7}$$

$$\lambda_3 \approx -\frac{t}{T} + \mathrm{O}(t^2) \qquad t = T_0 - T \lessgtr 0. \tag{5.8}$$

This teaches us that out of the symmetric states with small overlap, only the S_p^- (λ_3 is not relevant in this case) is stable near T_0 for $T < T_0$ independent of the bias structure. In particular the S_n^+ are unstable for $T > T_0$. We remark that there might exist S_n^+ with finite overlap at T_0 analogous to the Mattis state discussed above. They cannot be treated with the expansion (5.5).

Further results for the positive symmetric states have been obtained numerically and are summarized in figures 6–9. We concentrate on the lowest states $n = 2$ and $n = 3$ since they show the interesting features.

First of all these results are in agreement with what we have said before about their $T = 0$ behaviour and about bias zero. But even for $q = 2$ some new aspects, which we could not find in the literature, arise. While for $p = n$ and $a > 1/\sqrt{3}$ the upper limit of the stability region follows the curve $T_0 = 1 - a^2$, it turns out that for $p > n$ this upper limit goes down due to the effect of λ_3. Furthermore in a region around $a = \frac{1}{3}$, the symmetric state S_3^+ becomes unstable for small temperatures.

Comparing figures 3, 6 and 7 we see that, as a increases above $1/\sqrt{3}$, the spurious states get more weight for $q = 2$. This feature is also present but not so outspoken for

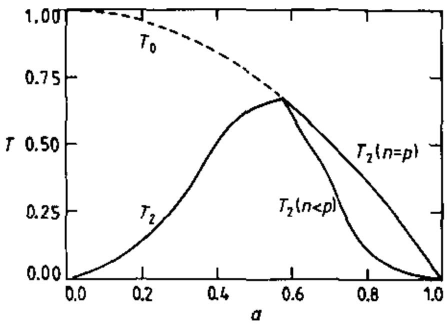

Figure 6. The temperatures T_0 and T_2 as a function of the bias for the Hopfield model.

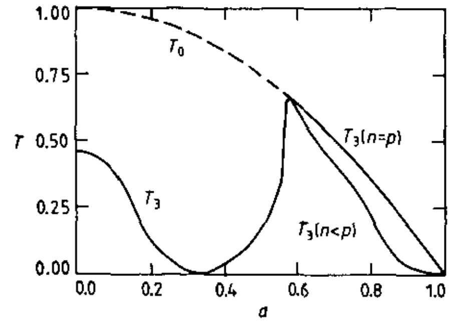

Figure 7. The temperatures T_0 and T_3 as a function of the bias for the Hopfield model.

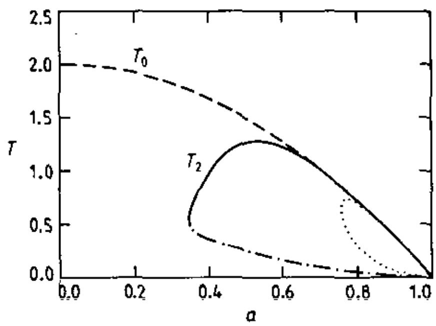

Figure 8. The temperatures T_0 and T_2 as a function of the bias for the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_1$ Potts network. The chain curve indicates the border of the stability region for $n \leq p$. The dotted curve encircles the region where the symmetric state is only stable for $n = p$.

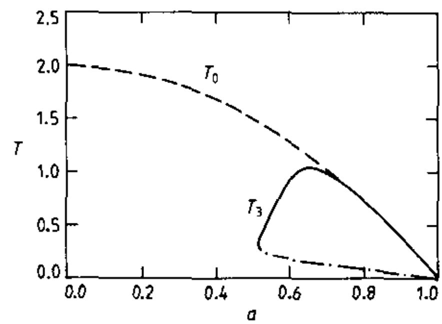

Figure 9. The temperatures T_0 and T_3 as a function of the bias for the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_1$ Potts network. The chain curve indicates the border of the stability region.

the $q = 3$ model with bias $\boldsymbol{B}_1$ (compare figures 4, 8 and 9). However for bias $\boldsymbol{B}_2$ it is not at all there (see figure 5). Furthermore we note that for $\boldsymbol{B}_2$ there is no influence from λ_3 on the S_2^+ and S_3^+ states.

For the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_2$ system one first encounters the Mattis state as one lowers the temperature T and this for the whole bias region. For the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_1$ system the only stable states for low bias up to $a = 0.35$ are the Mattis states. Again we remark that in this case the stability region for the $S_2^+(n < p)$ state is disconnected for high bias.

For both systems the stability region for higher symmetric states seems to get smaller as a function of a and n. So they perform better for low bias than the biased Hopfield model. This is even true for higher bias for the $q = 3$, $\boldsymbol{B} = \boldsymbol{B}_2$ system. Here one really finds a considerable temperature region depending on a where only the pure patterns are stable.

It will be interesting to see if the above features survive in the limit of saturation.

6. Concluding remarks

The main results of this paper are the following. We have studied Potts-glass models of neural networks with a finite number of biased patterns. In particluar the existence and stability properties of the Mattis states and symmetric states were discussed. The positive Mattis states are interesting since they are correlated with the learned patterns. The symmetric states correspond to mixtures of learned patterns and therefore they are spurious states of the network. Hence studying their (in)stability is important.

At zero temperature we have obtained the precise conditions for stability for an arbitrary number of Potts states q. In particular the positive Mattis states are always stable. Furthermore, they give complete retrieval. The negative Mattis states are stable for certain structures of the bias parameter. Also the stability results for the symmetric states depend very much on the bias structure. So for these cases we refer to the detailed explanation given in section 4. We have illustrated one of the main findings by applying these results to two typical $q = 3$ models. In the case that two states have equal probability to appear (the $\boldsymbol{B}_1$ system), we see that there are less stable spurious symmetric states than for models where all states have different probability (the $\boldsymbol{B}_2$ system). For the $\boldsymbol{B}_1$ system the free-energy levels cross for a certain value of a indicating that there are symmetric states which have a lower energy than the positive Mattis states. The latter have lower energy than the negative Mattis states.

For finite temperatures we have restricted ourselves mainly to $q \leq 3$ positive Mattis and the $n = 2, 3$ positive symmetric states. We have studied their stability properties in the whole temperature region as a function of the bias. First, for the $q = 3$ models there is a region (T_0, T_1) (see (5.1), (5.2)) where both a Mattis state and the zero solution are stable. From previous results [15] we know that at T_1 the overlap is discontinuous, e.g. for bias zero it jumps at $T_1 = 2.18$ from 0.75 to 0. This is in contrast to the Hopfield model where $T_1 \leq T_0$ and where the overlap is a continuous function of the temperature. Second, for certain values of the bias parameters, the temperature regions where the states of the network are stable can be disconnected. This has been seen for the $q = 2$ and $q = 3$ positive Mattis and positive symmetric states. Third, for positive n-symmetric states ($q = 2, 3$) the stability region for $n = p$ is smaller than or equal to the stability region for $n < p$ depending on the bias. Finally, as a increases the spurious symmetric states get more weight for $q = 2$, a feature that is less outspoken for the $q = 3$, $\boldsymbol{B}_1$ system. For the $q = 3$, $\boldsymbol{B}_2$ system there seems to be considerable temperature regions depending on a where only the pure patterns are stable. So in that sense one could conclude that the biased $q = 3$ models perform better than the biased Hopfield model.

Acknowledgments

We gratefully acknowledge stimulating discussions with R Dekeyser and H English.

Appendix. Behaviour of the eigenvalues λ_1 to λ_3 for $\beta \to \infty$

In the analysis of the stability of the states $S_n^{\pm}$ we consider two possibilities. Firstly, the eigenvalue λ_3 (equations (3.6) and (3.11)) exists ($n < p$). To calculate the limit

$\beta \to \infty$ of λ_3 we proceed as follows. We show that in this limit the average over the distribution of learned patterns $\{k^a\}$, $a = 1, ..., p$ of the expression

$$\frac{\sum_{\rho=1}^{q}(m_{k^{n+1},\rho} - ab_\rho)^2 U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)} - \left(\frac{\sum_{\rho=1}^{q}(m_{k^{n+1},\rho} - ab_\rho)U_\rho(n)}{\sum_{\rho=1}^{q} U_\rho(n)}\right)^2 \qquad \text{(A1)}$$

is positive or zero. Therefore we have to look carefully at the arguments of the different exponential functions occurring in the numerator and denominator of (A1) via $U_\rho(n)$ (3.8). Their value is determined in function of the Potts operator (2.2), the bias parameters B and the distribution of the first n patterns.

In more detail, for a given configuration $\boldsymbol{k} = \{k^1, ..., k^n\}$, the greatest arguments in the exponentials of $U_\rho(n)$ are obtained for a well defined set of ρ out of $\{1, ..., q\}$. Denoting this set by $A_{\boldsymbol{k}}$ and its cardinality by $\mathcal{N}(\boldsymbol{k})$ we get for the limit $\beta \to \infty$ of (A1)

$$\frac{\sum_{\rho\in A_{\boldsymbol{k}}}(m_{k^{n+1},\rho} - ab_\rho)^2}{\mathcal{N}(\boldsymbol{k})} - \left(\frac{\sum_{\rho\in A_{\boldsymbol{k}}}(m_{k^{n+1},\rho} - ab_\rho)}{\mathcal{N}(\boldsymbol{k})}\right)^2 . \qquad \text{(A2)}$$

If $\mathcal{N}(\boldsymbol{k}) > 1$ for some configuration $\boldsymbol{k}$, we rewrite (A2) as

$$\frac{1}{2[\mathcal{N}(\boldsymbol{k})]^2}\left\{\sum_{\substack{\rho,\sigma\in A_{\boldsymbol{k}}\\ \rho\neq\sigma}} \left[(m_{k^{n+1},\rho} - ab_\rho) - (m_{k^{n+1},\sigma} - ab_\sigma)\right]^2\right\} \geq 0. \qquad \text{(A3)}$$

For a configuration $\{\boldsymbol{k}, k^{n+1}\}$ with $\mathcal{N}(\boldsymbol{k}) > 1$ and such that $1 \neq k^{n+1} \in A_{\boldsymbol{k}}$ we find that (A3) is strictly positive. Hence $\lambda_3 \to -\infty$ as $\beta \to \infty$ and therefore we have instability.

When for all configurations $\boldsymbol{k}$ the cardinality $\mathcal{N}(\boldsymbol{k}) = 1$ and hence (A2) equals zero, we have that the limit $\beta \to \infty$ of the product of β and expression (A1) is zero since in this case (A1) decreases exponentially in β. This implies $\lambda_3 = 1$. Since the averages in (3.5) and (3.7) behave in the same way, we obtain that $\gamma_1 = 1$ and $\delta = 0$ in the limit $\beta \to \infty$. Hence, using (3.9) and (3.10) we see that $\lambda_1 = \lambda_2 = 1$. So, we have stability.

Secondly, the eigenvalue λ_3 does not exist ($n = p$). Looking at the expressions of λ_1 and λ_2 (equations (3.9) and (3.10)) we see that we have to analyze γ_1 (3.5) and δ (3.7). The calculation of γ_1 is similar to the one of λ_3 presented above. This leads to the following results.Firstly, if for all configurations $\boldsymbol{k}$, $\mathcal{N}(\boldsymbol{k}) = 1$ we have seen that $\gamma_1 = 1$ and $\delta = 0$. This means that $\lambda_1 = \lambda_2 = 1$ and hence we have stability. Secondly, when there exists a configuration $\boldsymbol{k}$ with $\mathcal{N}(\boldsymbol{k}) > 1$, we have two possibilities. If this configuration is such that $(m_{k^1,\rho} - ab_\rho)$ is not the same for all $\rho \in A_{\boldsymbol{k}}$, we have that $\gamma_1 < 0$ and this immediately implies that $\lambda_1 < 0$ or $\lambda_2 < 0$. So we have instability. However, when $(m_{k^1,\rho} - B_\rho)$ is the same for all $\rho \in A_{\boldsymbol{k}}$ for all configurations $\boldsymbol{k}$ with $\mathcal{N}(\boldsymbol{k}) > 1$, $\gamma_1 = 1$ and $\delta = 0$. Therefore we have stability.

We discuss two typical examples, e.g. S_{2l}^+ and S_{2l+1}^+ when there are two equal bias parameters. We use the form (4.1) for the bias parameters and we define $\boldsymbol{k}$ as in (4.7). First let us consider $n = 2l$ and take the specific configuration $\boldsymbol{k} = (k, ..., k, k+1, ..., k+1)$ where the first l components are equal to k and the last l components are equal to $k+1$. Then looking at the maximum over ρ of $(\sum_{a=1}^{n} m_{k^a,\rho} - nB_\rho)$ leads to $A_{\boldsymbol{k}} =$

$\{k, k+1\}$. According to the preceding analysis S^+_{2l} is unstable. Second, let us consider $n = 2l+1$, $q \geq 3$ and take the specific configuration $\boldsymbol{k} = (k, ..., k, k+1, ..., k+1, s)$ with the first l components equal to k, the following l components equal to $k+1$ and the last component $s \neq k, k+1$. Then the maximum over ρ of $(\sum_{a=1}^{n} m_{k^a,\rho} - nB_\rho)$ occurring in (A1) via $U_\rho(n)$ is equal to

$$\max\{lq - (2l+1) - (2l+1)ab_k, -2l + q - 1 - (2l+1)ab_s, \ -(2l+1) - (2l+1)ab_\rho\} \tag{A4}$$

where the different terms are coming from $\rho = k, k+1$, $\rho = s$ and other ρ. If both

$$\begin{aligned} &(l-1)q \geq (2l+1)a(b_k - b_s) \\ &\forall \rho \neq k, k+1, s: \ lq \geq (2l+1)a(b_k - b_\rho) \end{aligned} \tag{A5}$$

then $\{k, k+1\} \subset A_{\boldsymbol{k}}$ so $\mathcal{N}(\boldsymbol{k}) \geq 2$ and hence we have instability. The condition (A5) can be rewritten in the following way. If $k = 1$ we take $s = 3$ such that (A5) immediately gives the form (4.6). If $k \neq 1$ we take $s = 1$ such that the first condition in (A5) is trivially satisfied and the second one can be written as in (4.6). So under these conditions S^+_{2l+1} is unstable. Conversely, when (4.6) is not satisfied one can argue that $\mathcal{N}(\boldsymbol{k}) = 1$ for all configurations $\boldsymbol{k}$ (except for a finite number of values of a) and this implies stability.

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